Some Remark on Parallel Operators

L. Ceragioli

Interleaving Commutativity

Given $\mathcal{T}_1, \mathcal{T}_2$ transition systems, is the following true?

$$\mathcal{T}_1 \mathbin{|||} \mathcal{T}_2 = \mathcal{T}_2 \mathbin{|||} \mathcal{T}_1$$

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Yes! Any idea on how to prove it?

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Yes! Any idea on how to prove it?

(A little detail, it holds if $(s_1, s_2) = (s_2, s_1)$)

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Given $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ transition systems, is the following true?

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Yes! Any idea on how to prove it?

(A little detail, it holds if $((s_1, s_2), s_3) = (s_1, (s_2, s_3))$)

Synchronized Composition Associativity

Given $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ transition systems, and Syn_1, Syn_2 sets of actions, is the following true?

$$(\mathcal{T}_1 \parallel_{Syn_1} \mathcal{T}_2) \parallel_{Syn_2} \mathcal{T}_3 = \mathcal{T}_1 \parallel_{Syn_1} (\mathcal{T}_2 \parallel_{Syn_2} \mathcal{T}_3)$$

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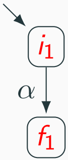
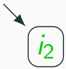
No!

Associativity Counterexample

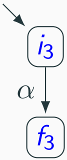
Transition System $\mathcal{T}_1$



Transition System $\mathcal{T}_2$



Transition System $\mathcal{T}_3$



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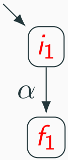
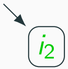


Transition System $\mathcal{T}_2$



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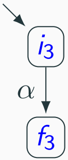

$Syn_1 = \{\alpha\}$ $\qquad$ $Syn_2 = \emptyset$

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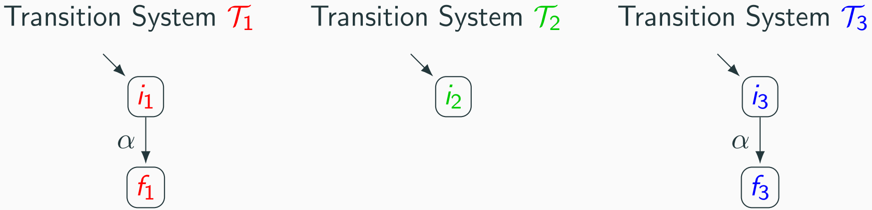

$Syn_1 = \{\alpha\}$ $\quad$ $Syn_2 = \emptyset$

$$(\mathcal{T}_1 \parallel_{Syn_1} \mathcal{T}_2) \parallel_{Syn_2} \mathcal{T}_3 \neq \mathcal{T}_1 \parallel_{Syn_1} (\mathcal{T}_2 \parallel_{Syn_2} \mathcal{T}_3)$$

Associativity Counterexample

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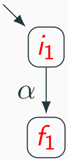
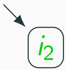
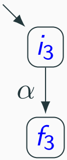


Transition System $\mathcal{T}_2$



Transition System $\mathcal{T}_3$



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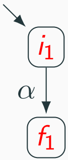

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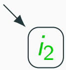

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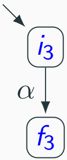
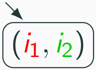


Transition System $\mathcal{T}_1 \parallel_{\{\alpha\}} \mathcal{T}_2$



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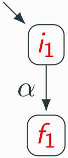
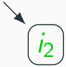


Transition System $\mathcal{T}_2$



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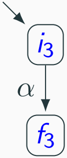

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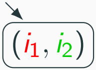
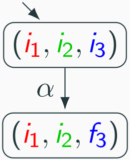


Transition System $(\mathcal{T}_1 \parallel_{\{\alpha\}} \mathcal{T}_2) \parallel_{\emptyset} \mathcal{T}_3$



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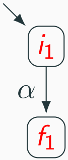
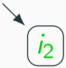


Transition System $\mathcal{T}_2$



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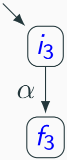

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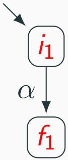

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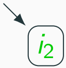

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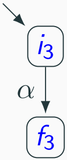
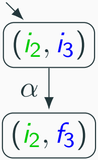


Transition System $\mathcal{T}_2 \parallel_\emptyset \mathcal{T}_3$



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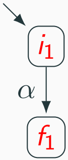

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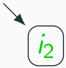

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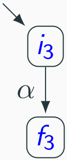

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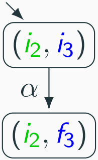
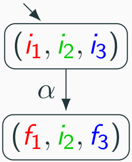


Transition System $\mathcal{T}_1 \parallel_{\{\alpha\}} (\mathcal{T}_2 \parallel_{\emptyset} \mathcal{T}_3)$



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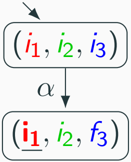
Transition System

$$(\mathcal{T}_1 \parallel_{\{\alpha\}} \mathcal{T}_2) \parallel_{\emptyset} \mathcal{T}_3$$



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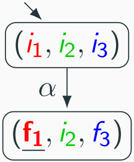

Synchronized Associativity: a special case

Given $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ transition systems, and *Syn* a set of actions, is the following true?

$$(\mathcal{T}_1 \parallel_{Syn} \mathcal{T}_2) \parallel_{Syn} \mathcal{T}_3 = \mathcal{T}_1 \parallel_{Syn} (\mathcal{T}_2 \parallel_{Syn} \mathcal{T}_3)$$

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Yes! Any idea on how to prove it?

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(recall that $Syn = Act_1 \cap Act_2$ is the implicit synchronization set of $\mathcal{T}_1 \parallel \mathcal{T}_2$, with Act_1 the actions of $\mathcal{T}_1$ and Act_2 the actions of $\mathcal{T}_2$)

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Yes! Any idea on how to prove it?

More on $\parallel$

Given $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ transition systems, it holds that:

$$(\mathcal{T}_1 \parallel \mathcal{T}_2) \parallel \mathcal{T}_3 = \mathcal{T}_1 \parallel (\mathcal{T}_2 \parallel \mathcal{T}_3) = \mathcal{T}_1 \parallel \mathcal{T}_2 \parallel \mathcal{T}_3$$

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Moreover:

$$\begin{gathered}\mathcal{T}_1 \parallel \mathcal{T}_2 \parallel \mathcal{T}_3 = \mathcal{T}_1 \parallel \mathcal{T}_3 \parallel \mathcal{T}_2 = \\ \mathcal{T}_2 \parallel \mathcal{T}_1 \parallel \mathcal{T}_3 = \mathcal{T}_2 \parallel \mathcal{T}_3 \parallel \mathcal{T}_1 = \\ \mathcal{T}_3 \parallel \mathcal{T}_1 \parallel \mathcal{T}_2 = \mathcal{T}_3 \parallel \mathcal{T}_2 \parallel \mathcal{T}_1\end{gathered}$$

More on $\|$

Composing the systems $\mathcal{T}_1, \ldots \mathcal{T}_n$ means that

- P_i can evolve independently with actions that are **only** in Act_i
- P_i, P_j can evolve by synchronizing with an action that is **only** in Act_i and in Act_j
- P_i, P_j, P_k can evolve by synchronizing with an action that is **only** in Act_i, Act_j and in Act_k
- $\ldots$

The composed system of $\mathcal{T}_1, \ldots \mathcal{T}_n$ is such that, for each action α, a state of the obtained transition system can perform an α action if and only if all the processes P_i with $\alpha \in Act_i$ perform the α action.